Interpolating between ferromagnets and spin glasses

E. Eisenstein, I. Kanter, and D. A. Kessler
Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel
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The nature of some classical N-spin systems with N^β independent random interactions is examined for $0 \leq \beta \leq 2$. It is found that in the mean-field limit the appropriate scaling of the interactions in the leading order is $N^{1-\beta/2}$ for $\beta \leq 1$ and $\sqrt{N}$ for $1 < \beta \leq 2$, consistent with both known limiting cases, ferromagnets ($\beta = 0$) and spin glasses ($\beta = 2$). The dynamics is found to be spin-glass-like for $\beta > 1$, whereas for $\beta < 1$ the dynamics is characterized by short relaxation times. Similar results are obtained for a one-dimensional Ising system where the strength of the random interaction decays with the distance r as $r^{-\beta}$.
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The two main classes of spin systems which have been exhaustively investigated analytically and numerically and which continue to be a source of new concepts in statistical mechanics are ordered and disordered classical spin systems [1, 2]. These classes are typified by ferromagnetic systems and spin-glass systems, respectively. A prototypical Hamiltonian for these systems is

$$H = -\sum_{(ij)}^{N} J_{ij} S_i S_j, \tag{1}$$

where S_i is a spin variable and N is the number of spins that is assumed to be in the thermodynamic limit. The spin variables that we will consider here are either defined by a global constraint as in the spherical model $\sum S_i^2 = N$ [3, 4] or are of Ising type $S_i = \pm 1$. The form of the interactions, the matrix J_{ij}, identifies the topology of the system; the dimensionality, short-range or long-range interactions, etc. In the following discussion, we mainly concentrate on the mean-field limit where all elements of the matrix J_{ij} are of the same order.

For the prototype of ordered systems in this limit, the mean-field Ising ferromagnet (FM), all interactions are equal to a unique positive value, whereas for the prototype of disordered systems, spin glasses, each interaction is an independent random variable which can take positive or negative values. More precisely, in order for the free energy to be an extensive quantity the interactions in the FM case should scale as

$$\tilde{J}/N \qquad \text{(FM)}, \tag{2}$$

where $\tilde{J}$ is a positive constant of order one, and in the spin-glass (SG) case the interactions should scale as

$$\tilde{J}/\sqrt{N} \qquad \text{(SG)} \tag{3}$$

where $\tilde{J}$ in this case is a random variable with zero mean and unit variance [5]. Hence the number of quenched random variables in these cases scales as N^β where $\beta = 0$ for the FM case and $\beta = 2$ for the SG case. The origin of the classification of ordered and disordered systems comes from the nature of the low-temperature phase (or the ground state). In the ground state of the Ising FM or related models [6], spins are completely correlated to their interactions since no frustration [7] is present, whereas for SG's the ground state has zero magnetization but there is no simple way to predict its structure for a particular realization [1, 2].

The main goal of this paper is to examine systems consisting of only

$$N^\beta, \qquad 0 \leq \beta \leq 2, \tag{4}$$

independent random interactions, denoted later as J_1, J_2, ..., J_{N^β} [8]. For simplicity, we assume that each one of the random interactions appears with the same probability. Hence each one of the interactions appears on average $N^{2-\beta}$ times in the matrix J.

We are motivated in this study by a number of different considerations. The first arises from the study of finite-dimensional random spin systems. Whereas most of the analytical work to date has been carried out only in the mean-field (infinite-dimensional) limit, the reality is of course finite dimensional. One way to come closer to this situation is by reducing the number of independent interactions, thus mimicking the fact that in the finite-dimensional system, the finite connectivity coupled with the geometrical constraints reduces the number of independent interactions to a value much less than the $O(N^2)$ associated with the mean-field limit.

A second motivation derives from the study of neural networks. A sequential updating of a system of the type described by Eq. (1), with $\beta < 1$ can be shown [9] to be equivalent to a particular perceptron. In broad terms (see Ref. [9] for details), the architecture is a simple perceptron consisting of $N-1$ binary input units and 1 binary output. The $N-1$ weights are chosen to be symmetrical around the center weight, $W_i = W_{N-i}$. The updating of the output unit corresponds to the updating of a single spin under zero-temperature Glauber dynamics. Each successive input to the perceptron is constructed by a shift of the previous inputs by one unit to the right, and the assignment of the last output to the leftmost input. One can show that the dynamics of this perceptron is equivalent to the zero-temperature dynamics of the spin

system [Eq. (1)] under sequential updating with the interactions given by $J_{ij} = W_{i-j \bmod N+1}$.

Many questions may be asked regarding this class of models described by Eqs. (1)–(2). However, the fundamental question is whether there is universal behavior as a function of β, the number of independent random interactions, or do the physical properties depend on the details of the randomness? In the discussion below, the following three properties are examined: (1) What is the appropriate scaling of the interactions? (2) What is the nature of the ground state; i.e, whether it belongs to the class of ordered systems or to that of disordered systems. (3) Is the dynamical behavior characterized by short relaxation times as for FM or is it characterized by exponentially long relaxation times as for SG?

In the following discussion we distinguish between different ensembles of matrices constructed from N^β random variables, although the scaling results are the same, since the details of the results is different.

The first case to be examined is $\beta = 1$ with spherical spin variables and where the interactions are only a function of the "distance" between pairs of spins and where periodic boundary conditions are assumed

$$J_{ij} = G_{|i-j| \bmod N/2}. \tag{5}$$

The number of independent interactions for an even N, for instance, is $N/2$, G_1, ..., $G_{N/2}$, and note that the diagonal elements of the matrix J are zero. The probability distribution $P(G_l)$ is taken to be Gaussian with zero mean and unit variance $\chi_l^{-1} = 1$, except for the boundary term $\chi_{N/2}^{-1} = 2$. The matrix J with spatial structure, Eq. (5), is known as a Toeplitz matrix (TM). Its eigenvalues λ can be labeled by a Fourier index l, $0 \le l \le N/2$, and are doubly degenerate for $0 < l < N/2$

$$\lambda_l = 2\sum_{k=1}^{N/2} \chi_k G_k \cos\left[\frac{2\pi}{N} kl\right]. \tag{6}$$

The joint probability distribution of the λ_l is given by

$$P(\lambda_0, \ldots, \lambda_{N/2}) \propto \exp\left[-\frac{1}{N}\sum_{l=0}^{N/2} \chi_l \lambda_l^2\right] \delta\left(\sum_{l=0}^{N/2} \chi_l \lambda_l\right). \tag{7}$$

where again the boundary term $\chi_0 = 1/2$. In the case of spherical spin variables the ground-state energy is equal to minus the largest eigenvalue of the matrix J, $E_{gs} = -\lambda_{max}$. A straightforward calculation [10] shows that the maximal eigenvalue in our case is

$$\lambda_{max} \propto \sqrt{N \ln N}. \tag{8}$$

Since the Hamiltonian, Eq. (1), is proportional to the interaction strength, the appropriate rescaling of the interactions in order for the free energy to be an extensive quantity is $1/\sqrt{N \ln N}$. Note that the reduction of β from 2 to 1 affects the scaling of J only by a logarithmic correction. The result, Eq. (8), was confirmed in simulation on systems of sizes $10 - 10^5$, see Fig. 1.

One way that the case $\beta < 1$ can be implemented is by assuming that the interactions are a function of the distance modulo $N^{1-\beta}$ between any pair of spins

$$J_{ij} = G_{|i-j| \bmod N^{1-\beta}}. \tag{9}$$

The structure of the first row of the matrix J_{1j}, for instance, in this case is a "window" of length N^β which is repeated up to $N/2$ and the second half of this row is a mirror image of the first half around $N/2$, since periodic boundary conditions are assumed. The structure of the eigenvalues in this case is given by

$$\lambda_l = 2\sum_{k=1}^{N^\beta} \chi_k G_k \sum_{m=0}^{\frac{1}{2}N^{1-\beta}} \cos\left[\frac{2\pi}{N}(k + N^\beta ml)\right], \tag{10}$$

where it is clear that the number of times each one of the random G_k appears is equal to the number of windows, $N^{1-\beta}$. In order to calculate the scaling of the largest eigenvalue one may use the following observation. The second argument of the cos term, in Eq. (10), is independent of k and it contributes a constant phase only when $l = qN^{1-\beta}$, where $q = 1, 2, \ldots, \frac{1}{2}N^\beta$. Therefore, for these eigenvalues the second sum of Eq. (10) gives a number of order $N^{1-\beta}$ and the first sum then is a random walk of N^β steps where the size of each one of them is of order $N^{1-\beta}$. Hence, each one of these N^β eigenvalues scale as $N^{1-\beta/2}$. In a manner parallel to the derivation of Eq. (8), one can then verify that the maximal eigenvalue among these eigenvalues scales as

$$\lambda_{max} \sim N^{1-\beta/2}\sqrt{\ln N}. \tag{11}$$

It is also clear that this eigenvalue is the maximal one among the N eigenvalues, since for the rest of the eigenvalues the cos does not have a constant phase for each k. Then the phase is essentially random so that $\lambda \propto \sqrt{N}$. Thus, for such systems that consist of N^β random inter-

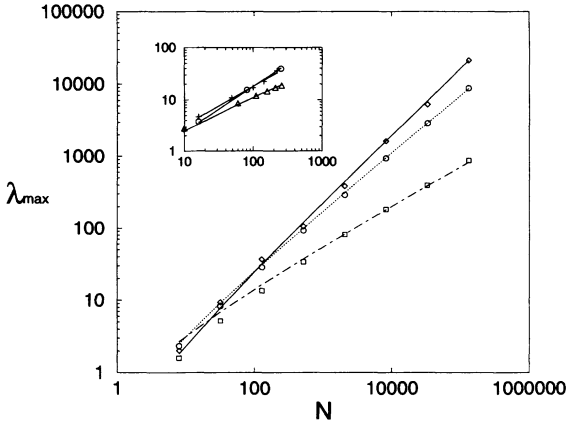

FIG. 1. The largest eigenvalue as a function of N for Toeplitz matrix with windows of size N^β, $\beta = 1$ (□), 0.5 (○), and 0.25 (◇). The equation of the lines are given by: dotted line $0.56N^{0.75}\sqrt{\ln N}$, dot-dashed line $N^{0.5}\sqrt{\ln N}$, solid line $0.31N^{0.875}\sqrt{\ln N}$. Inset: The largest eigenvalue as a function of N for the unstructured case for $\beta = 1$ (△), 1/2 (+), and 0.25 (○). The curves are $0.78\sqrt{N \ln N}$, $0.43N^{0.75}$, $0.32N^{0.875}$, respectively.

actions of type Eq. (5) and spherical spins, the appropriate rescaling is to leading order $1/N^{1-\beta/2}$. Results of numerical diagonalization of matrices up to size 10^5 and for various values of β are presented in Fig. 1. Let us only comment that the above-mentioned results for $\beta \leq 1$ are not affected when the bias of the interactions is subtracted such that $\sum G_k = 0$.

The above is to be contrasted with another implementation of the case $\beta < 1$, namely to construct the symmetric matrix J at *random* from the N^β independent random variables G_k. For each J_{ij} $(i < j)$ one of the G is chosen at random with equal probability. Hence, the average number of times that each interaction appears in the unstructured matrix is $N^{2-\beta}$ to leading order. Numerical diagonalization of such matrices up to $N = 1000$ indicates that the average maximal eigenvalue scales as $N^{1-\beta/2}$ as for the previous case, see inset of Fig. 1. To understand this, assume that the randomness of the matrix J is restricted such that in each row of the matrix each interaction appears the same number of times, $N^{1-\beta}$. One can easily verify that one of the eigenvectors is (1, 1,...., 1) with corresponding eigenvalue

$$\sum_{l=1}^{N} G_l = N^{1-\beta} \sum_{l=1}^{N^\beta} G_l = O(N^{1-\beta/2}) . \quad (12)$$

This eigenvalue is on the average one half of the time positive, since the sets $\{G_k\}$ and $\{-G_k\}$ have the same probability. The rest of the eigenvalues, however, scale as $\sqrt{N}$, since they are insensitive to the bias. Hence, for the one half of the time where the bias $\sum G_k$ is positive, the largest eigenvalue scales as $N^{1-\beta/2}$ with a FM ground state. Note that since there is only one eigenvalue of this order there is no $\sqrt{\ln N}$ factor in this case. In simulations on unstructured matrices (where our argument is only approximate) it was found that there always exists a FM eigenvector with structure $(1+\epsilon_1, 1+\epsilon_2, \ldots, 1+\epsilon_N)$, where $|\epsilon_i|$ is less than one and is a decreasing function of N. Note that, in contrast to the window implementation above, the scaling $\lambda_{\max} \propto N^{1-\beta/2}$ is a direct result of the bias of the G and that the scaling of the largest eigenvalue reduces to $\sqrt{N}$ as for the SG case if the bias is subtracted. Nevertheless, if the bias is not removed, the scaling $\lambda_{\max} \propto N^{1-\beta/2}$ obtains in both implementations.

A crucial question is whether this scaling is relevant also to discrete spin variables such as Ising spins, where $E_{gs} \neq -\lambda_{\max}$. We examined this question by using zero-temperature Monte-Carlo dynamics initiated from many random configurations, where the ground-state energy was chosen as the minimal energy among the metastable states. For each size of the system, a halting criterion was chosen by monitoring the rate of appearance of new metastable states (MS) with the result that the number of MS grows exponentially for $N < 50$ [11]. The results for $\beta = 1$, 1/2, and 1/4, for instance, are presented in Fig. 2 and indicate that indeed even for Ising systems the interactions should scale as $N^{\beta/2-1}$ to achieve an extensive energy.

Let us now try to examine the structure of the ground-state energy as a function of β and try to estimate the probability that it is characterized by a FM order. Let us

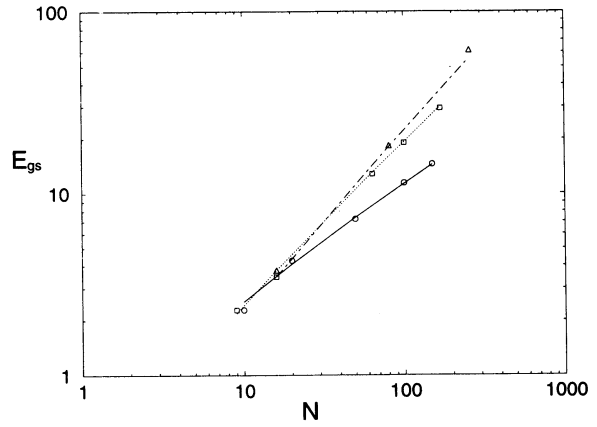

FIG. 2. The ground-state energy E_{gs} for the Ising case with an unstructured matrix. $\beta = 1$ (○), $\beta = 0.5$ (□), $\beta = 0.25$ (△). The equation of the lines are: dot-dashed line $0.28N^{0.875}\sqrt{\ln N}$, dotted line $0.43N^{0.75}\sqrt{\ln N}$, solid line $0.8N^{0.5}\sqrt{\ln N}$.

first examine the case of spherical spins. As we discussed above, for the TM J constructed from windows of size N^β, the number of eigenvalues of $O(N^{1-\beta/2})$ is N^β and their Fourier indices are $l = qN^{1-\beta}$, $q = 1, \ldots, \frac{1}{2}N^\beta$. Numerical diagonalization of such matrices indicates that the probability for the ground state to be of wave number l is $N^{-\beta}$, independent of l. Hence, the probability for the ground state to be ferromagnetic, $l = 0$, is proportional to the inverse of the number of eigenvalues of $O(N^{1-\beta/2})$. A similar behavior was observed in simulations of the same matrix J but with Ising spins, where the Fourier spectrum of the ground state consists of one dominant peak. However the results in this case are noisy since the largest examined N is much smaller. Note that for the unstructured matrix case the probability for a FM ground state is 1/2, independent of β, as was discussed above.

Besides the rescaling of the interactions, an interesting question is the nature of the dynamics as a function of β. Is the dynamics at finite temperature governed by short relaxation times as for FM systems, or is it governed by long relaxation times as for the SG case? A direct answer to this question requires a careful analysis of finite-temperature Monte-Carlo simulations of random systems, which in general, if possible, is a heavy numerical task. Therefore, a different measure was used to shed light on this question. This measure is the probability, P_{gs}, that starting from random initial condition the system ends at the ground state under zero-temperature Glauber dynamics. For FM systems P_{gs} is expected to be of order one, independent of N, whereas for SG systems P_{gs} is expected to scale exponentially small with N [11]. This behavior is deduced from the fact that for FM systems there is only a finite number of metastable states whose basin of attractions are comparable with that of the ground state, whereas for SG systems the effective number of such metastable states is exponential with N. In our simulations for $N \leq 50$, the number

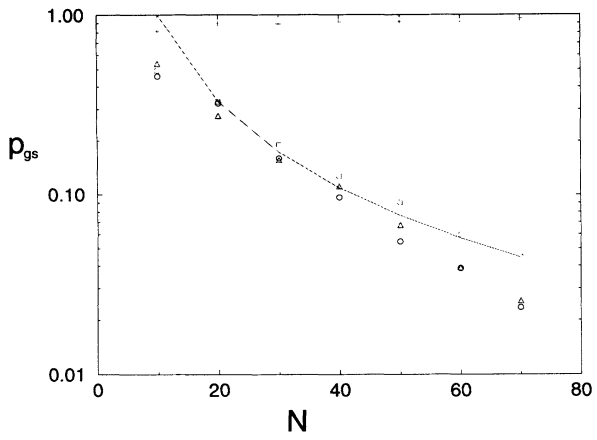

FIG. 3. The probability to evolve into the ground state from random initial conditions as a function of N for $\beta = 2$ (○), $\beta = 1.5$ (△), $\beta = 1$ (□), $\beta = 0.5$ (+). The line is proportional to $N^{-1.6}$.

of random initial conditions was greater than the number of metastable states, and hence the ground state is expected to be reached. Furthermore, the number of trials before the ground state was first reached was verified to be much smaller than the number of random initial conditions. The results of our simulations on Ising spins and an unstructured matrix J indicate that the region $0 \leq \beta \leq 2$ is divided into the following three regimes. (a) $0 \leq \beta < 1$, P_{gs} is of order one. (b) $\beta = 1$, $P_{gs} \propto N^{\gamma}$ where $\gamma \approx -1.6$. (c) $1 < \beta \leq 2$, $P_{gs} \propto e^{-AN}$ with $A \sim 0.05$ independent of β, see Fig. 3. Hence, the behavior of the system for $\beta > 1$ is expected to be similar to SG systems $\beta = 2$, both from the aspect of the rescaling of the interactions and from the long relaxation times that characterize the dynamics. At $\beta = 1$ a transition from exponentially slow relaxation to power law relaxation is observed. Finally, note that for a TM J with $\beta \leq 1$, P_{gs} was observed to be independent of N as for regime (a).

We have seen that one way of interpolating between FM and SG is to allow the G from which the TM is constructed to be correlated over large "distances". Another way to accomplish this is to choose

$$\langle G_i \rangle = 0, \qquad \langle G_i G_j \rangle = f(|i-j|), \tag{13}$$

where

$$f(i) \sim \frac{1}{i^{\beta}}, \qquad i \gg 1 \tag{14}$$

Using Eq. (6) the scaling properties of the eigenvalues of such a "colored" TM are

$$\langle \lambda_k^2 \rangle = 4 \sum_{i,j=0}^{N/2} \chi_i \chi_j f(|i-j|) \cos\left(\frac{2\pi i k}{N}\right) \cos\left(\frac{2\pi j k}{N}\right)$$
$$\sim N^{2-\beta}/k^{1-\beta}, \qquad 1 \ll k \ll N \tag{15}$$

Since λ_k falls off with k, we expect that the maximal eigenvalue will have a fairly small value of k. Then, since the λ scale with N as $N^{1-\beta/2}$, we get that the maximal eigenvalue scales the same way [without the $\ln^{1/2}(N)$ factor, since the number of large eigenvalues does not increase with N]. Numerical results of our simulations, using a Fourier representation [12] of the G to generate the colored TM, are in excellent agreement with our scaling predictions [10].

Finally, let us mention a few open questions regarding the interpolating systems between FM and SG which certainly deserve further research. It will be interesting to find an analytical method to calculate the free energy and then to examine the behavior as a function of β at finite temperature, especially the critical exponents at the transition to the low-temperature phase. The probability distribution of the eigenvalues as a function of β and the gap between the maximal eigenvalue and the next one are examples of general features of correlated random matrices which still need to be addressed.

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